AIAA JOURNAL
Vol. 33, No. 11, November 1995

Criterion for the Detachment of Laminar and Turbulent Boundary Layers

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A criterion is proposed for the detachment of incompressible, steady, two-dimensional boundary layers under the influence of adverse pressure gradients on smooth, faired surfaces, in the absence of heat or mass transfer. It is proposed that detachment will occur if a suitably defined dimensionless parameter measuring the ratio of the boundary-layer momentum defect to the freestream momentum reaches a maximum value. The approach requires knowledge of the relations between integral thicknesses during separation, but no empirically determined critical value is required for any boundary-layer shape factor. Consensus values of such parameters follow accurately from the proposed procedure. The criterion is exact in the context of the Kármán–Pohlhausen theory and precisely satisfied by numerically evaluated solutions of the Falkner–Skan equation. The criterion is consistent with Simpson's experimental data and compares well with several established semi-empirical methods for predicting the intermittent transitory detachment of turbulent boundary layers.

Nomenclature

H = shape factor, δ^*/θ
H_1 = Head's entrainment parameter
h = shape factor, $(H-1)/H$
I = momentum-flux/mass-flow ratio, Eqs. (1) and (2)
J = profile parameter in Kármán–Pohlhausen theory
L = arbitrary distance from wall, $L > \delta$
m = pressure gradient parameter in Falkner–Skan equation
R = equilibrium velocity profile constant
Re_δ = Reynolds number based on displacement thickness, $(U_e\delta^*)/\nu$
U = time mean velocity component parallel to wall
U_s = scaling velocity in Schofield–Perry theory
x, y = distance coordinates parallel and normal to wall
γ_p = fraction of time the flow moves forward
δ = boundary-layer thickness associated with y location where $U/U_e = 0.995$
δ^* = boundary-layer displacement thickness
Λ = boundary-layer shape factor, δ^*/δ
η = normalized position coordinate normal to wall, y/δ
η_s = normalized position coordinate normal to wall, for Schofield–Perry theory
θ = boundary-layer momentum thickness
ν = fluid kinematic viscosity
ρ = fluid density
σ = specific momentum defect, Eq. (3)

Subscripts

d = associated with detachment in laminar boundary layers or with intermittent transitory detachment in turbulent layers
$*$ = associated with maximum in $\sigma(h)$ function
e = at boundary-layer edge (at $y = \delta$)
δ = associated with entire boundary layer
0 = at the zero velocity streamline of separation bubble

Received Sept. 16, 1994; revision received April 25, 1995; accepted for publication May 1, 1995; presented as Paper 95-2152 at the AIAA 26th Fluid Dynamics Conference, San Diego, CA, June 19–22, 1995.

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Introduction

FLOW separation is the single most important factor that limits the efficiency and operating range of fluid dynamic devices, and it justifiably occupied a large share of fluid mechanics research for many decades. Much progress has been made, but an ability to predict the occurrence of separation in all circumstances of interest still has not been reached and the exploration of the physics of both the laminar and turbulent separation processes continues. An extensive review of the status of this field was given recently by Simpson.[1]

In the case of steady mean flows, the term separation was originally associated with the location where the wall shear stress vanishes, $\tau_w = 0$. Later on, based on work initiated by Kline and expanded by numerous other authors, a more precise terminology was developed to describe such events.[1] Separation was redefined to describe the entire region that includes reversed flow and displays significant velocity components normal to the wall. The location of $\tau_w = 0$ is designated as the location of detachment, at least for laminar boundary layers. Turbulent separation is a much more complex process that occurs gradually over a finite streamwise distance and requires further elaboration. At any location inside the separation, turbulent fluctuations cause the flow to alternate between forward and backward directions. The fraction of time for forward flow (positive instantaneous velocity component in the x direction) in the laminar sublayer is designated as γ_p. Upstream of the separation γ_p is unity and gradually declines over the separation to values quite small compared with unity. The location where γ_p near the wall is equal to 0.99 is called the location for incipient detachment, whereas the $\gamma_p = 0.8$ and 0.5 values are associated with intermittent transitory detachment (ITD) and transitory detachment (TD), respectively. The location where the time-averaged wall shear stress is zero is called the detachment, without any qualifier. Observations show that for all practical purposes detachment and TD coincide. For further details, the reader is referred to Simpson's review.[1]

The fundamental detachment criteria are thus stated in terms of wall shear stress and/or intermittency. Neither of these is easy to measure or compute, and there is a need for more easily manageable indications of the occurrence of separation. From a practical point of view, it is desirable to obtain such an indication from the least possible amount of information. Past research has shown that certain integral properties of the velocity profile just before separation constitute sufficient input to determine, at least approximately, whether separation occurs. The occurrence of detachment has been related by various authors to specific values of certain boundary-layer profile or shape parameters. These criteria are secondary to the fundamental ones given in terms of intermittency or shear stress, but they are nevertheless very useful. They are especially important in the context of integral theories of boundary layers, which tend to break down near detachment and need the additional assist from the empiricisms embodied in separation criteria. A thorough review of correlations and criteria dealing with separation was given by Kline et al.[2]

Present day computational fluid dynamics (CFD) methods are capable of predicting the details of separated flows and the need for detachment criteria diminished correspondingly. However, CFD prediction of separated flows still requires highly sophisticated codes and powerful computers, whereas integral theories offer fast and in many cases quite adequate solutions with the use of minimal human and computer resources. They remain the methods of choice in many industrial situations dominated by cost and time constraints.

The original motivation for this work was educational, aimed at constructing a physically plausible, "back-of-the envelope" style explanation for the reasonable success of the generally used simple detachment criteria. The inquiry uncovered the existence of a maximum in a particular function that describes the evolution of the boundary layer in terms of integral parameters. The occurrence of the maximum correlates with the occurrence of detachment in both laminar and turbulent separating boundary layers to a remarkable degree and may be used as a detachment criterion. The paper presents this correlation and its applications to six separate descriptions of boundary layers.

The considerations of this paper are limited to incompressible, two-dimensional boundary layers with steady mean flow. The boundaries are streamlined, smooth surfaces, and curvature, heat transfer, or mass transfer effects are assumed to be absent.

Physical Arguments

The ideas offered in this paper grew out of the intuitive expectation that boundary layers exposed to adverse pressure gradients will detach (separate) if the momentum carried by the boundary layer becomes less than some fraction of the freestream momentum. In the following we treat this expectation as a hypothesis, cast it into a quantitative form, and compare the predictions to established theoretical and semiempirical results.

A reasonable point of departure is to consider the behavior of the ratio of the boundary-layer and freestream momentum fluxes. Since the latter depends on the choice of the freestream streamtube to be considered, we shall deal with values of momentum per unit mass flow for both regions, a quantity we shall call specific momentum.

Assuming a uniform freestream, the momentum per unit mass flow is readily shown to be equal to the edge velocity, independently of the width of the streamtube considered:

$$I_e = \frac{\int_\delta^L \rho U^2 \,\mathrm{d}y}{\int_\delta^L \rho U \,\mathrm{d}y} = U_e \tag{1}$$

For the boundary layer, simple integration and application of the usual boundary-layer thickness definitions result in

$$I_\delta = \frac{\int_0^\delta \rho U^2 \,\mathrm{d}y}{\int_0^\delta \rho U \,\mathrm{d}y} = U_e\left[1 - \frac{\theta}{\delta - \delta^*}\right] \tag{2}$$

The ratio of specific momentum of the boundary layer to that of the freestream is given by the expression in the bracket on the right-hand side of Eq. (2). A somewhat simpler expression is obtained if we consider deviations (defects) from the freestream values, in the spirit of integral theories. The ratio of the specific momentum defect for the boundary layer to the freestream specific momentum can be expressed as

$$\frac{I_e - I_\delta}{I_e} = \frac{\theta}{\delta - \delta^*} \equiv \sigma \tag{3}$$

The parameter σ, called here the normalized specific momentum defect, is a measure of how much less specific momentum the boundary layer has than the freestream. The expectation is that separation will occur if σ is somehow large. The largeness of σ will be clarified by the investigation of available data. However, first a few comments are in order on dimensionless parameters used to characterize velocity profiles and on integral thicknesses of boundary layers.

Integral Thicknesses and Profile Parameters

The preceding arguments involved three integral boundary-layer thicknesses: δ, δ^*, and θ. These quantities may be used to form various dimensionless ratios, but only two of the ratios can be chosen independently: all others can be expressed as functions of the chosen two. A commonly used such ratio is the shape factor H, defined as

$$H = \frac{\delta^*}{\theta} \tag{4}$$

Kline introduced a completely equivalent but more convenient shape factor[2] defined as

$$h = \frac{\delta^* - \theta}{\delta^*} = 1 - \frac{1}{H} \tag{5}$$

The parameter h has the range from 0 to 1, and in some cases it leads to simpler formulas than H does. Numerous experiments show that h monotonically increases in the vicinity of detachment,[2] which suggests the use of h as an independent variable. This choice permits the consideration of boundary-layer evolution in the detachment zone without having to deal explicitly with dependencies on streamwise position.

Another shape factor in frequent use is

$$\Lambda = \frac{\delta^*}{\delta} \tag{6}$$

The parameter σ is connected to the shape factors H, Λ, and h through the relationships

$$\sigma = \frac{\Lambda}{(1 - \Lambda)H} \tag{7}$$

and

$$\sigma = \frac{\Lambda}{1 - \Lambda}(1 - h) \tag{8}$$

A quantity equal to the reciprocal of σ ($H_1 = 1/\sigma$) has been introduced by Head and Patel[3] as an important parameter in an integral method based on considerations of entrainment. Developments related to Head's parameter will be utilized in this paper.

If a velocity profile is specified analytically, then the integral thicknesses and their ratios can be readily calculated. Analytical specifications typically include one or more dimensionless profile parameters that are constants for a given profile and vary as functions of x. Attached boundary layers (in which inertial, pressure, and viscous effects all play important roles) may require two parameters for accurate description,[4] but there is strong evidence that near separation (where the dynamics are dominated by inertial and pressure forces whereas viscous forces are unimportant) one parameter is sufficient.[2,5] Since this paper focuses on the separation process, all boundary-layer models used in the following are of the one-parameter variety.

A single-parameter family description implies that the velocity profiles evolve through a unique sequence of (nondimensional) shapes as the boundary layer approaches and goes through the process of separation.

Given a single-parameter velocity profile, the integral thicknesses can be computed readily. The thicknesses and their ratios will be functions of the profile parameter, i.e., $\delta^*/\delta(=\Lambda)$ and θ/δ both will be functions of the parameter built into the profile definitions. The same is true of any other dimensionless parameter formed from the ratios, such as σ and h. Elimination of the profile parameter between the so-constructed Λ and h functions gives a $\Lambda(h)$ relation that can be combined with the generally valid Eq. (8) to yield a single $\sigma(h)$ function. This function defines a single curve on the $\sigma(h)$ plane that represents the locus of the states through which the boundary layer passes while approaching and undergoing separation.

Should the boundary-layer profile contain two parameters, then the same procedure would lead to a family of curves on the $\sigma(h)$ plane. Clearly, the applicability of a single-parameter description is a major simplification.

Proposed Criterion

The criterion proposed here evolved gradually through the study of a number of velocity profile models and empirical correlations. However, this paper can be structured more logically if the criterion is stated in the beginning and justification is given subsequently. The criterion is stated as follows:

Detachment occurs where the normalized specific momentum defect σ reaches a maximum as a function of the shape factor h, i.e., where the following condition is satisfied:

$$\left(\frac{d\sigma}{dh}\right)_* = 0 \tag{9}$$

Since h increases monotonically through separation, the x location of detachment is determined through the h_* value selected by the criterion. The streamwise location for detachment is unambiguous in laminar layers. In the case of a turbulent boundary layer, the empirical evidence shows that the preceding criterion corresponds to intermittent transitory detachment.

Each of the sources used here (theories, correlations, data) identifies detachment in some fashion. The identification is not always in the form of specifying a value of h, but it is possible to find a value of h associated with ITD, designated as h_d, using the methodology of the particular source. We shall refer to h_d as the "actual" detachment value, to be compared with the h_* values derived using the proposed criterion.

Since σ and h both can be expressed in terms of other parameters, this criterion may be cast in many forms. We retain Eq. (9) as a form closest to a physical interpretation. In the rest of this paper we shall apply this criterion to both laminar and turbulent boundary layer models.

Laminar Boundary Layers

Kármán–Pohlhausen Theory

The Kármán–Pohlhausen (KP) theory[6] describes boundary-layer profiles as a one-parameter family of fourth-order polynomials:

$$\frac{U}{U_e} = \left(2 + \frac{J}{6}\right)\eta + \frac{J}{2}\eta^2 - \left(2 - \frac{J}{2}\right)\eta^3 + \left(1 - \frac{J}{6}\right)\eta^4 \tag{10}$$

The parameter J is a nondimensional, streamwise velocity gradient defined as

$$J = \frac{\delta^2}{\nu}\frac{dU_e}{dx} \tag{11}$$

The parameter σ is easily computed and found to be

$$\sigma = \frac{5328 - 48J - 5J^2}{378(84 + J)} \tag{12}$$

The streamwise evolution of σ is simulated by considering it to be a function of the shape factor h, which is expected to grow monotonically with x. It is easy to express h as a function of J as

$$h = 1 - \frac{5328 - 48J - 5J^2}{378(36 - J)} \tag{13}$$

The $\sigma(h)$ relation, as defined parametrically by Eqs. (12) and (13), is shown in Fig. 1. The assumptions built into the theory limit its applicability to the range $-12 < J < 12$. $J = -12$ corresponds to detachment, i.e., to a zero slope at the wall, and $J > 12$ corresponds to profiles in which the velocity within the boundary layer exceeds the edge value, a situation not possible without curvature effects that are not considered here.

The paramount feature of the KP plot in Fig. 1 is the existence of a maximum, which is readily shown to occur where $J = -12$, i.e., the maximum coincides exactly with detachment. The values of other relevant variables at the detachment are easily calculated:

$$h_* = 5/7 \cong 0.714 = h_d$$
$$\sigma_* = 4/21 \cong 0.191 \tag{14}$$

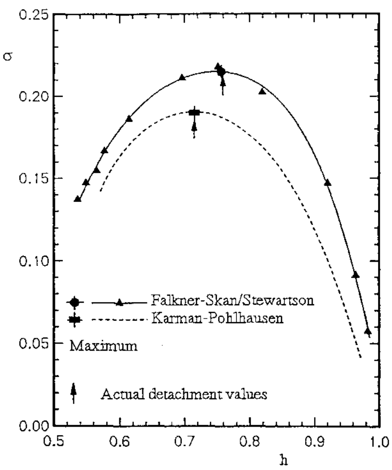

Fig. 1 $\sigma(h)$ correlation for laminar boundary layers. Symbols on the left side of the line type designate points of maximum σ. Full triangles designate values numerically computed from the Falkner–Skan theory,[9] and the continuous line is a fourth-order fit to the computed points. Vertical arrows indicate h_d values for the case to which they point.

Stewartson's Solution to the Falkner–Skan Equation

The Falkner–Skan (FS) equation describes laminar boundary layers for a class of external velocity distributions in which the edge velocity varies as the mth power of x. Stewartson gave a family of solutions[7] of this equation for negative values of m, i.e., for adverse pressure gradients. Reversed flow regions exist near the wall for $m < -0.091$.

We may view the separation process as an evolution of the boundary layer through a sequence of profile shapes, all of which are members of the same family. The unseparated solutions of the Falkner–Skan equation, combined with the separated profiles of Stewartson, constitute a smoothly varying sequence of profiles that may be used to describe the entire laminar separation process. (This approach was successfully applied to supersonic boundary layers by Lees and Reeves.[8]) Using the boundary-layer thicknesses computed for these profiles (data by Rogers[9] were used), the $\sigma(h)$ relation can be readily constructed and is presented in Fig. 1. The curve is not quite smooth, presumably due to numerical inaccuracies, but the presence of a maximum is clearly evident. The parameter values at maximum, given below, are those possessed by a fourth-order polynomial least-squares fit to the points shown:

$$h_* = 0.749$$
$$\sigma_* = 0.215 \tag{15}$$

The h value for the separation profile ($m = -0.091$) is $h_d = 0.752$, very close to the value at the σ maximum location given earlier. (Each vertical arrow in Fig. 1 indicates the h_d values associated with the curve to which it points.) Since numerical solutions are not available for neighboring values of m, it cannot be stated with certainty that the maximum of σ coincides exactly with detachment. In any case, it is very close.

Considering that the KP theory postulates a very simple approximate profile, and that equating the Stewartson family of profiles to a sequential development in a single flow is only an assumption, the agreement between the KP and the FS results must be considered quite good.

Turbulent Boundary Layers

Since there are no exact analyses available for turbulent boundary layers, the proposed criterion is tested using various empirical descriptions designed for the description of turbulent boundary layers through the separation process.

Coles Wall–Wake Profile

The Coles wall–wake (CWW) profile[10] has been demonstrated to apply to a wide range of attached turbulent boundary layers. It is also a reasonable description for a turbulent boundary layer in various

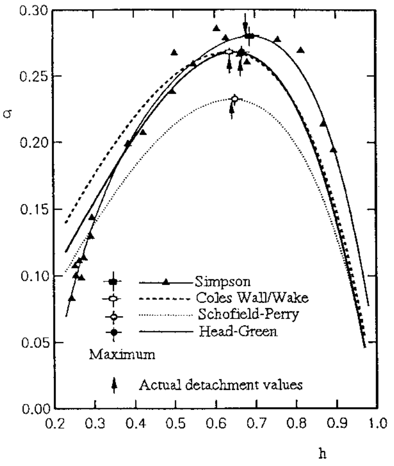

Fig. 2 $\sigma(h)$ correlation for turbulent boundary layers. Symbols on the left side of the line type designate points of maximum σ. The solid line is a fourth-order fit to data of Simpson et al.[15] Vertical arrows indicate h_d values for the case to which they point.

stages of the separation process, as long as certain sign conventions are obeyed concerning the friction velocity.[2]

The CWW profile is a two-parameter family. The two parameters may be chosen in various ways, the most useful choice being such that one parameter reflects the viscous effects and the other corresponds to inertial effects. One such possible parameter pair is Re_{δ^*} and Λ. It has been shown by Kline et al.[2] that the $h(\Lambda, Re_{\delta^*})$ function is only very weakly dependent on the Reynolds number. Furthermore, the dependence continues to weaken as the boundary layer approaches detachment and almost completely disappears after detachment. Since our interest is focused on detachment itself, we shall neglect the weak Reynolds number dependence and consider h to be a function of Λ only. Kline et al.[2] have shown that the $h(\Lambda)$ correlation has a very simple form,

$$h = R\Lambda \tag{16}$$

where R is an empirical constant equal to 1.5. Combining Eq. (16) with Eq. (8), we get

$$\sigma = \frac{h(1-h)}{R-h} \tag{17}$$

Experimental data support this relation in the range $0.2 < h < 0.95$, which includes attached flow and all stages of the separation process.[2] As illustrated in Fig. 2, the relation has a similar appearance to the curves that apply to laminar boundary layers. In particular, Eq. (17) has a maximum at

$$h_* = R - \sqrt{R(R-1)} \cong 0.634$$
$$\sigma_* = \left(\sqrt{R} - \sqrt{R-1}\right)^2 \cong 0.268 \tag{18}$$

The h_* value obtained from the proposed criterion agrees very closely with the $h_d = 0.63$ value recommended by Kline as a criterion for ITD.

Schofield–Perry (SP) Correlation

Perry and Schofield[11] and Schofield[12] gave a one-parameter expression for turbulent velocity profiles, originally designed to describe boundary layers subject to adverse pressure gradients, as follows:

$$\frac{U}{U_e} = 1 - \frac{U_s}{U_e}\left[1 - 0.4\sqrt{\eta_s} - 0.6\sin\left(\frac{\pi}{2}\eta_s\right)\right] \tag{19}$$

where

$$\eta_s = \frac{y - y_0}{\delta}$$

The profile parameter is U_s/U_e, related to the previously defined parameters by

$$\frac{U_s}{U_e} = 2.86\Lambda \tag{20}$$

Schofield[13] has shown later that the profile also applies resonably well to detached boundary layers, provided the origin of the y coordinate axis is moved from the wall to the zero-mean velocity streamline of the separation bubble (this is reflected by the y_0 term in the nondimensional coordinate). Since $y_0(x)$ is not generally known, the profiles are defined in the detached region only within an unknown shift in the y direction. Equation (19) is intended to describe the outer layer. Closer to the wall the sine term is negligible, and the description is replaced by a purely square root dependence. Very close to the wall (corresponding to the very thin logarithmic region and the linear viscous sublayer), the equation is not valid. By extensive comparison to experimental results, the authors concluded that ITD will occur if $U_s/U_e \cong 1.10$ (Fig. 11 in Ref. 13).

In possession of Eq. (19), the integral thicknesses and their dimensionless ratios can be evaluated without difficulty. Since the expression is not valid close to the wall, the integrations were carried out from $y = 0.02\delta$ to δ, as done by the authors, and $y_0 = 0$ was used. The resulting $\sigma(h)$ function is

$$\sigma = \frac{h(1.053 - h) - 0.053}{1.55 - h} \tag{21}$$

which is, not surprisingly, similar in form to Eq. (17), derived for the CWW profile. The variable values at the maximum are

$$h_* = 0.643$$
$$\sigma_* = 0.232 \tag{22}$$
$$\left(\frac{U_s}{U_e}\right)_* = 1.13$$

The h_d value associated with the SP profile for $U_s/U_e = 1.10$ is 0.638. The SP formulation thus leads to the same behavior, the same conclusions, and very nearly the same numerical values as the CWW description.

Entrainment Formulas by Head and Green

The entrainment method of Head and Patel[3] requires a relation between the integral parameters H_1 and H. This relation is readily rewritten into the $\sigma(h)$ function needed for the application of the present criterion. The $H_1(H)$ relations experienced a historical development (see review by Lock[5]). The early forms, developed by Head, were based on attached flow information and led to a $\sigma(h)$ function that had a maximum value only in the physically meaningless limit of $h \to 1$. However, a later version, developed by Green,[14] is applicable to separated flows. This correlation is given as follows:

$$H_1 = 2 + 1.5\left(\frac{1.12}{H-1}\right)^{1.093} + 0.5\left(\frac{H-1}{1.12}\right)^{1.093} \tag{23}$$

for $1.3 \le H \le 4$. The equivalent $\sigma(h)$ function is

$$\sigma = \left\{2 + 1.5\left[\frac{1.12(1-h)}{h}\right]^{1.093} + 0.5\left[\frac{h}{1.12(1-h)}\right]^{1.093}\right\}^{-1} \tag{24}$$

for $0.23 \le h \le 0.75$. Equation (24), plotted in Fig. 2, clearly possesses a maximum, a fact noted by Young,[4] although he cautioned against attributing too much significance to this observation. The parameter values at the maximum are

$$h_* = 0.649$$
$$\sigma_* = 0.268 \tag{25}$$
$$H_{1*} = 3.732$$

No specific h_d value is suggested by these authors, but the results agree closely with the CWW and SP results.

Table 1 Comparison of actual and predicted values of h

Source	h_d	h_*	$h_* - h_d$	σ_*
Kármán–Pohlhausen	0.714	0.714	0.000	0.191
Falkner–Skan	0.752	0.749	−0.003	0.215
Coles	0.63	0.634	+0.004	0.268
Schofield–Perry	0.638	0.643	+0.005	0.232
Head–Green	0.643	0.649	+0.006	0.268
Simpson	0.638	0.685	+0.047	0.280

Experimental Data by Simpson et al.

The detailed experimental data of Simpson et al.,[15] which were extensively used by both Kline et al. and Perry and Schofield, are also shown in Fig. 2. The general trend of the data clearly shows the existence of the maximum, despite the data scatter. A fourth-order polynomial, least-squares fitted to the data, yields the following values for the maximum:

$$h_* = 0.685 \qquad \sigma_* = 0.280 \tag{26}$$

The value of h_d (as determined from measured intermittency data) is 0.638. The agreement, although not as good as those found with the previous correlations, is still within the precision level expected of turbulent flow data.

Uncertainty Analysis

Table 1 offers a comparison of the actual h_d and the h_* values. The table shows that the two quantities agree within less than 1% in all cases except Simpson's, where the two differ by about 7%.

It should be born in mind that the uncertainty involved in the experimental determination of the location of intermittent separation is significant and that this uncertainty is reflected in the h_d values. (The uncertainties of velocity profile measurements also affect h_d, but they are much less significant.) None of the sources give uncertainties associated with h_d, but the data scatters suggest that two sigma limits for a single experiment are around ±15% or greater. Correlations are expected to perform somewhat better than individual experiments, since they may be viewed as a form of ensemble averaging over flows of the same family and therefore carry a correspondingly reduced uncertainty level. In view of the large uncertainty levels of the data, the agreement shown in Table 1 is quite remarkable.

The proposed correlation, when applied to a particular family of profiles, predicts the actual detachment value of h_d associated with that family quite accurately. In the two laminar cases, the agreement between the respective h_d and h_* values is significantly better than the agreement between the h_d values predicted by the two models. In the turbulent cases, the agreement between the h_d and h_* values is about as good as the variation in h_d among the four sources.

Comments

Maximum Occurs in Parameter Space

It is important to point out that the maxima of the $\sigma(h)$ and $\sigma(x)$ functions are not equivalent. Both σ and h are functions of x, and the $\sigma(h)$ function may be stated as a parametric function of x. We have assumed that the parameter h increases monotonically across the detachment point and have shown that the parameter reaches a maximum at detachment.

Each streamwise location corresponds to a representative point on the $\sigma(h)$ curve. As the boundary layer approaches separation, the representative point approaches the maximum of σ. If detachment does occur, then the point passes the maximum and continues to move towards increasing h and decreasing σ values. In this case, there is a proper maximum at detachment in both $\sigma(x)$ and $\sigma(h)$, but not in $h(x)$.

It is also possible, however, that the boundary layer approaches detachment, but before it could actually detach, the pressure gradient changes to a favorable value, the shape factor h diminishes again, and the representative point reverses the direction of its movement on the $\sigma(h)$ curve. In this case there will be a maximum both in $\sigma(x)$ and $h(x)$, but not in $\sigma(h)$, since the latter is indeterminate:

$$\frac{d\sigma}{dh} = \left(\frac{d\sigma}{dx}\right) \bigg/ \frac{dh}{dx} = \frac{0}{0} \tag{27}$$

It follows that the existence of a maximum in the $\sigma(x)$ relation is a necessary but not sufficient condition of detachment.

Streamwise Location of Detachment

In the possession of h_*, the streamwise location of ITD is determined from the $h(x)$ function describing the flow. Since $h(x)$ monotonically increases through separation, the location is unambiguously identified. The particular values of σ_* therefore do not play any role in identifying the location of detachment.

The σ_* values are nevertheless of interest. The average σ_* value is 0.20 for laminar layers and 0.26 for turbulent layers, implying that the specific momentum of an attached boundary layer is at least 80% of the freestream value for laminar flow and 74% for turbulent flows. Since turbulent layers are known to be more resistant to separation, the difference should not be surprising.

Theoretical Justification Needed

The coincidence between detachment and the satisfaction of Eq. (9) is mathematically exact in case of the KP theory and practically exact for the FS and CWW cases (for ITD). The high precision of the coincidence was unexpected and provides a strong stimulus for serious thought. The intuitive arguments that led to Eq. (9) were not expected to produce the remarkable agreements shown in Table 1. It is not unreasonable to anticipate that Eq. (9) might be the consequence of a more general relationship that is merely illustrated by the specific cases studied here. Attempts to find such a broader basis have not been successful.

Summary

It has been demonstrated that for the class of boundary layers investigated, the normalized specific momentum defect reaches a maximum value at detachment, when considered a function of the shape parameter h. In the case of laminar boundary layers, the location of detachment is well defined. In the case of turbulent boundary layers, the maximum σ condition corresponds to ITD. This behavior was found to occur in two approximate analytical descriptions of laminar boundary layers, in three different empirical correlations describing turbulent boundary layers near detachment, and in Simpson's experimental data for a separated flow. The agreement is very accurate for laminar flows and well within the general scatter range of the turbulent correlations studied. A plausibility argument was given as to why this behavior is reasonable, but no solid theoretical basis can be offered.

The coincidence of the maximum σ condition and the occurrence of detachment suggests that the existence of the former can be used as a criterion for detachment. A satisfying feature of this criterion is that empirically determined critical values are not required for any parameter (such as $H = 2.7$ at ITD for a turbulent boundary layer). If a one-parameter empirical or analytical profile definition is available, then the proposed criterion offers a simple method to determine the ITD value of the profile parameter.

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